Random matrix ensembles from nonextensive entropy

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The classical Gaussian ensembles of random matrices can be constructed by maximizing Boltzmann-Gibbs-Shannon's entropy, $S_{\mathrm{BGS}}=-\int d\mathbf{H}[P(\mathbf{H})]\ln[P(\mathbf{H})]$, with suitable constraints. Here, we construct and analyze random-matrix ensembles arising from the generalized entropy $S_q=\{1-\int d\mathbf{H}[P(\mathbf{H})]^q\}/(q-1)$ (thus, $S_1=S_{\mathrm{BGS}}$). The resulting ensembles are characterized by a parameter q measuring the degree of nonextensivity of the entropic form. Making $q\to 1$ recovers the Gaussian ensembles. If $q\neq 1$, the joint probability distributions $P(\mathbf{H})$ cannot be factorized, i.e., the matrix elements of $\mathbf{H}$ are correlated. In the limit of large matrices two different regimes are observed. When $q<1$, $P(\mathbf{H})$ has compact support, and the fluctuations tend asymptotically to those of the Gaussian ensembles. Anomalies appear for $q>1$: Both $P(\mathbf{H})$ and the marginal distributions $P(H_{ij})$ show power-law tails. Numerical analyses reveal that the nearest-neighbor spacing distribution is also long-tailed (not Wigner-Dyson) and, after proper scaling, very close to the result for the 2×2 case — a generalization of Wigner's surmise. We discuss connections of these "nonextensive" ensembles with other non-Gaussian ones, such as the so-called Lévy ensembles and those arising from soft confinement.

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I. INTRODUCTION

The Gaussian ensembles of random matrix theory provide the standard statistical description of spectral fluctuations in a multiplicity of quantum systems ranging from nuclei to disordered mesoscopic conductors and classically chaotic systems [1–5].

Gaussian ensembles can be obtained from two postulates: the invariance of the joint distribution probability $P(\mathbf{H})$ with respect to changes of bases and the statistical independence of matrix elements. An alternative and more appealing way of constructing random matrix ensembles uses a maximum entropy principle [1,6]. One constraint is normalization

$$\int d\mathbf{H}\, P(\mathbf{H}) = 1. \tag{1}$$

The other one has the purpose of confining the spectrum, but is otherwise arbitrary (as long as the integral converges)

$$\int d\mathbf{H}\, P(\mathbf{H})\mathrm{tr}\, V(\mathbf{H}) = 1 \tag{2}$$

(the trace ensures rotational invariance). For instance, the Gaussian ensemble of real symmetric matrices is obtained by the simplest choice

$$V(\mathbf{H}) = \mathbf{H}^2. \tag{3}$$

It has been proven that, in the limit of large matrices, and for a strong enough confining potential V, local fluctuation properties tend to those of the Gaussian case, whatever the shape of V [7,8].

To escape from Gaussian universality one must consider soft-confinement potentials [9,10], or breaking rotational invariance. The latter case typically arises when matrix elements H_{ij} are identically distributed and independent (non-Gaussian) random variables. For instance, Cizeau and Bouchaud constructed anomalous "Lévy ensembles" by drawing H_{ij} from a long-tailed distribution [11]. Furthermore, if one allows matrix elements to depend on the indices ij, a huge variety of ensembles emerges, with a behavior different from Gaussian, e.g., "banded matrices" [12].

The purpose of this paper is to present a new way of constructing non-Gaussian ensembles while preserving rotational invariance. The idea is to use a maximum entropy approach with the usual constraints but with the nonextensive entropy [13]

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$$S_q[P(\mathbf{H})] = \frac{1 - \int d\mathbf{H}[P(\mathbf{H})]^q}{q-1}, \tag{4}$$

where q is a free, real parameter ($q=1$ recovers Shannon's standard entropy). This scheme produces a variety of ensembles, with q controlling the degree of confinement. Some ensembles belong to the Gaussian universality class but others exhibit anomalous behavior, characterized by distributions having power-law tails.

The explicit construction of these q ensembles is presented in Sec. II, where we also derive expressions for marginal distributions and the joint density of eigenvalues. Remarkably, for large matrices, the q ensembles can be represented as a superposition of Gaussian ensembles. This allows us to obtain closed analytical formulas for the eigenvalue density, level-spacing probability distributions, etc. (Sec. III). The comparison of analytical results with numerical simulations is the subject of Sec. IV. We present in Sec. V the concluding remarks.

II. THE GENERALIZED ENSEMBLES

For simplicity we restrict our analysis to ensembles of real and symmetric matrices $\mathbf{H}$ — extensions are straightforward. The volume element in this space is

$$d\mathbf{H} = \prod_{i=1}^{N} dH_{ii} \prod_{i<j}^{N} dH_{ij}, \tag{5}$$

where it is understood that matrices are of size $N \times N$. Generalized ensembles are obtained by maximizing the entropy of Eq. (4) subjected to normalization, Eq. (1), and

$$\frac{\int d\mathbf{H}\, \mathrm{tr}\, \mathbf{H}^2 [P(\mathbf{H})]^q}{\int d\mathbf{H}[P(\mathbf{H})]^q} = \sigma^2, \tag{6}$$

with σ a constant having units of energy (we are assuming that of $\mathbf{H}$ is a Hamiltonian). Equation (6) is the generalization of the usual constraint that leads to the Gaussian ensembles in the standard maximum entropy approach. Arguments justifying the use of the escort probabilities P^q, and applications of this generalized maximum entropy scheme to various problems, can be found in Ref. [13].

Using the Lagrange multiplier technique, it is straightforward to find the distribution of maximum entropy

$$P(\mathbf{H}) \propto \exp_q(-\lambda\ \mathrm{tr}\, \mathbf{H}^2), \tag{7}$$

where we have defined the q-exponential function [13]

$$\exp_q(x) \equiv \{[1 + (1-q)x]_+\}^{1/(1-q)}, \tag{8}$$

with

$$\cdots_+ = \max\{\cdots, 0\} \tag{9}$$

[note that $\exp_1(x) = \exp(x)$]. The omitted normalization constant in (7) and the parameter λ can be determined from the constraints (1) and (6). (Some preliminary results along these lines have been obtained by Evans and Michael [14].)

The ensemble defined by Eq. (7) will be called the "q-orthogonal ensemble" (qOE), as it can be seen in (7) that the probability distribution depends only on $\mathrm{tr}\, \mathbf{H}^2$, an orthogonal invariant. When $q \to 1$ the q-exponential function tends to the usual exponential, and one recovers the Gaussian orthogonal ensemble (GOE). Except for the $q=1$ case, the q exponential in (7) *cannot* be factorized into a product of (marginal) distributions for individual matrix elements H_{ij}, which are then correlated. We can already verify that the cases $q<1$ and $q>1$ are qualitatively different. Equations (8) and (9) show that for $q<1$ the distributions have compact support; if $q>1$, there are always power-law tails (we are assuming $\lambda > 0$; see below).

To proceed with the analysis of qOE it will be convenient to think of matrices $\mathbf{H}$ as points in a d-dimensional Euclidean space [1,16,17]. The first N components of a point $\mathbf{r}$ correspond to diagonal elements H_{ii}, the last ones to the upper triangle H_{ij}, $i<j$

$$\mathbf{r} = (H_{11}, \ldots, H_{NN}, \sqrt{2}H_{12}, \ldots, \sqrt{2}H_{N-1,N}). \tag{10}$$

The dimension of this space equals the number of independent matrix elements of $\mathbf{H}$, i.e.,

$$d = \frac{N(N+1)}{2}. \tag{11}$$

The scaling of H_{ij} by $\sqrt{2}$ makes the probability distribution (7) spherically symmetric in R^d, i.e., $P_{\mathrm{qOE}}(\mathbf{r})$ is the product of a uniform distribution in the angles, and a radial distribution [15]

$$\mathcal{P}(r; q, \sigma, N) \propto r^{d-1} \exp_q(-\lambda r^2), \tag{12}$$

where

$$r^2 \equiv \mathbf{r} \cdot \mathbf{r} = \mathrm{tr}\, \mathbf{H}^2. \tag{13}$$

The observations above imply that qOE belongs to the wider category of "spherical ensembles" recently studied by Le Caër and Delannay [16,17].

For $q>1$, the distribution (12) has a power-law tail which scales as $1/r^{(1+\mu)}$ with

$$\mu = \frac{2}{q-1} - d. \tag{14}$$

Then, the normalization condition cannot be satisfied for all values of q, but only by those making $\mu > 0$, i.e.,

$$-\infty < q < 1 + \frac{2}{d}. \tag{15}$$

(Note the formal similarity between this problem and the generalized random walker in d dimensions [18].)

The Lagrange multiplier λ is given by

$$\lambda = \frac{1}{\sigma^2}\frac{d}{2-d(q-1)}. \tag{16}$$

Inside the region (15) (i.e., normalizability) λ is always positive.

Integrating Eq. (7) over all variables but one, we obtain the marginal distributions for diagonal and off-diagonal matrix elements [19]

$$P(H_{ii}) \propto \exp_{q'}(-\lambda' H_{ii}^2), \tag{17}$$

$$P(H_{ij}) \propto \exp_{q'}(-2\lambda' H_{ij}^2), \tag{18}$$

where

$$q' = \frac{2-(d-3)(q-1)}{2-(d-1)(q-1)}, \tag{19}$$

and

$$\lambda' = \frac{d}{2\sigma^2}\frac{2-(d-1)(q-1)}{2-d(q-1)}. \tag{20}$$

The following properties can be easily verified. The parameter q' is an increasing function of q, and around the critical value $q=1$ one has

$$q' = q + O((q-1)^2). \tag{21}$$

In addition, λ' is always positive. Then, in parallel with the global $P(\mathbf{H})$, the marginal distributions also decay as power laws or have compact support, depending on q being larger or smaller than 1, respectively. We remark that the matrix elements are not independent, so $P(\mathbf{H})$ cannot be reconstructed from the marginal probabilities (17) and (18).

The joint density of eigenvalues can be obtained in a straightforward way [4,14]

$$P(\varepsilon_1, \ldots, \varepsilon_N) \propto \prod_{i<j=1}^{N} |\varepsilon_j - \varepsilon_i| \exp_q\left(-\lambda \sum_{i=1}^{N} \varepsilon_i^2\right). \tag{22}$$

The part that is responsible for level repulsion is identical to that in GOE because it arises only from orthogonal symmetry. The difference is in the confinement term, which in the present case is a nonseparable q exponential. Thus, the "potential" that confines the spectrum is not a single-particle quadratic well, as in GOE. It is rather a mean field, proportional to the moment of inertia $\Sigma\varepsilon_i^2$.

We can get a clear view of the generalized ensembles by noting that these are connected to the so-called fixed-trace ensembles (FTE) and, for large N, to the Gaussian ensembles. In fact, recall that FTE are defined by [1,16,17,20]

$$P_{\mathrm{FTE}}(\mathbf{H}; r, N) \propto \delta(\mathrm{tr}\,\mathbf{H}^2 - r^2). \tag{23}$$

Let $f(\mathbf{H})$ be an arbitrary function and consider the averages in both ensembles qOE and FTE, namely

$$\langle f(\mathbf{H})\rangle_{\mathrm{qOE}}(q,\sigma,N) = \int d\mathbf{H}\, P_{\mathrm{qOE}}(\mathbf{H}; q,\sigma,N) f(\mathbf{H}), \tag{24}$$

and

$$\langle f(\mathbf{H})\rangle_{\mathrm{FTE}}(r,N) \int d\mathbf{H}\, P_{\mathrm{FTE}}(\mathbf{H}; r,N) f(\mathbf{H}). \tag{25}$$

Then, we have the relation

$$\langle f(\mathbf{H})\rangle_{\mathrm{qOE}}(q,\sigma,N) = \int_0^\infty dr\, \mathcal{P}(r; q,\sigma,N)\langle f(\mathbf{H})\rangle_{\mathrm{FTE}}(r,N). \tag{26}$$

The average over qOE can be calculated in two stages. First, do the average over the angles, for a fixed radius r. This corresponds to an FTE average. Then, average over radii, with the weighting function $\mathcal{P}(r)$. Of course, the same is true for the GOE, which corresponds to the particular case $q=1$. The relationship between qOE (or GOE) and FTE is analogous to that between the canonical and microcanonical ensembles of statistical mechanics.

Equation (26) involves no approximations. Although exact, it is not very useful because it requires the knowledge of fixed-trace averages. However, if one is interested in the limit of large matrices, important simplifications can be made.

III. THE LARGE-*N* LIMIT

The key point is that, for N large enough, the FTE average in the right-hand side of (26) can be approximated by an average in a GOE having the property $\langle \mathrm{tr}\,\mathbf{H}^2\rangle = r^2$. Then, if we know the GOE average of a given function, its corresponding qOE average can in principle be calculated by doing just one integration. We will analyze in detail two spectral statistics: the eigenvalue density

$$\rho(\varepsilon; q,\sigma,N) = \left\langle \sum_{i=1}^{N} \delta(\varepsilon - \varepsilon_i) \right\rangle, \tag{27}$$

and the distribution of level spacings

$$p(s; q,\sigma,N) = \langle \delta(\varepsilon_{i+1} - \varepsilon_i - s)\rangle. \tag{28}$$

In the last equation, ε_i and ε_{i+1} are two consecutive eigenvalues lying at the center of the band, i.e., $\varepsilon_i \approx 0$ [29]. It is (or will become) clear that other statistics, e.g., two-point correlation functions, can also be considered along the same lines.

In order to obtain the qOE averages of (27) and (28), we need the corresponding FTE expressions to be further averaged with $\mathcal{P}(r; q,\sigma,N)$, as indicated by Eq. (26). However, we will approximate FTE averages by the corresponding GOE ones. Then, the basic ingredients become the "semicircle law" (for the eigenvalue density)

$$\rho(\varepsilon; N, r) = \frac{N^2}{2\pi r^2}\sqrt{\frac{4r^2}{N} - \varepsilon^2}, \tag{29}$$

and Wigner's surmise

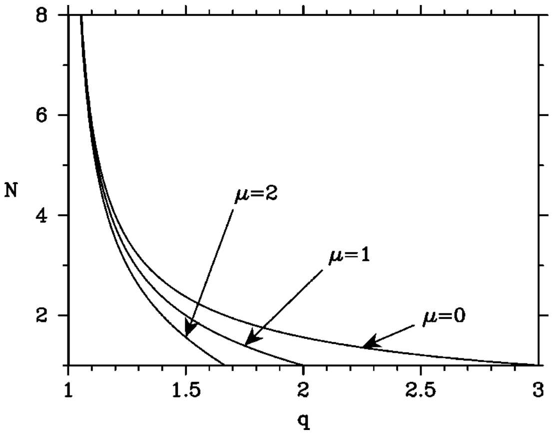

FIG. 1. When $q>1$ the ensembles qOE lie in the region limited by the axes and the curve $\mu=0$ (normalization frontier). As N becomes large, the maximum q allowed tends to 1. Lines correspond to families of ensembles having the same power-law tails (labeled by μ).

$$p(s;N,r) = \frac{N^3 s}{2\pi r^2}\exp\left(-\frac{N^3 s^2}{4\pi r^2}\right), \tag{30}$$

giving the level-spacing distribution. Equations (29) and (30) are good approximations for both GOE and FTE distributions when N is large [16].

We recall that if $q>1$, the normalization condition (15) limits the value of N to a finite domain. On the other hand, the case $q<1$ does not present such a problem. So, we analyze each case separately.

A. Ensembles with $q<1$

Except for providing an energy scale, σ plays no special role. From now on, without loss of generality, we set $\sigma=1$. If desired, σ can be restored at any time by dimensional analysis.

When $N\to\infty$ (q fixed) the radial distribution of qOE tends to

$$\mathcal{P}(r;q,N) \propto r^{d-1}[1-r^2]^{1/(1-q)}, \tag{31}$$

limited to the domain $0\leqslant r\leqslant 1$. As d grows, the distribution is squeezed against $r=1$, being concentrated in a small region below $r=1$, of width $O(1/d)=O(1/N^2)$. It can be verified that both the level density (29) and the spacing distribution (30), when considered as functions of r, have widths which are $O(1)$. Thus, the radial distribution is much narrower and we can safely approximate

$$\mathcal{P}(r;q<1,N\to\infty) \simeq \delta(r-1). \tag{32}$$

We conclude that, when $q<1$ and $N\to\infty$, the ensembles qOE tend to the GOE [as far as it concerns the distributions being studied, namely Eqs. (27) and (28)].

B. Ensembles with $q>1$

When $q>1$ the possible ensembles are restricted to a region in the plane q–N that gets thinner as $N\to\infty$ (see Fig. 1). The natural coordinates in this region are N (or d) and μ [see Eq. (14)], the latter controlling the tails of $\mathcal{P}(r)$ and other distributions. For instance, substituting (14) and (16) into (17) or (18), one immediately verifies that the marginal distributions behave asymptotically as

$$P(H_{ij}) \sim \frac{1}{H_{ij}^{1+\mu}}. \tag{33}$$

The radial distribution (12), as a function of r,μ,N, becomes

$$\mathcal{P}(r,\mu,N) \propto r^{d-1}\left[1+\frac{d}{\mu}r^2\right]^{-(d+\mu)/2}. \tag{34}$$

This expression allows the identification of some well-known ensembles as special members of the qOE class: the Cauchy-Lorentz ensemble corresponds to $\mu=1$. An integer $\mu>1$ produces Student's ensembles (see Refs. [16,17] and references therein; see also Ref. [22]).

Now, we analyze the limit $N\to\infty$ while keeping $\mu>0$ fixed, i.e., we move upwards along the curves of Fig. 1. As in the case $q<1$, examined before, there is a limiting distribution. Some simple algebra leads to

$$\mathcal{P}(r,\mu,N\to\infty) \propto r^{-(1+\mu)}\exp\left(-\frac{\mu}{2r^2}\right). \tag{35}$$

Only when $\mu\to\infty$, $\mathcal{P}$ tends to the delta function (32), and GOE is recovered. For finite μ the width of $\mathcal{P}(r)$ is at least $O(1)$. In any case, the average of a given GOE distribution with $\mathcal{P}(r)$ gives the corresponding qOE distribution [via Eq. (26) with $\langle f(\mathbf{H})\rangle_{\text{FTE}}\sim\langle f(\mathbf{H})\rangle_{\text{GOE}}$, when $N\to\infty$]. Let us first consider the density of states. Inserting Eqs. (29) and (35) into (26), we obtain

$$\rho(\varepsilon;\mu) \propto \int_{\sqrt{N}\varepsilon/2}^{\infty} dr \frac{\sqrt{4r^2-N\varepsilon^2}}{r^{\mu+3}}\exp\left(-\frac{\mu}{2r^2}\right). \tag{36}$$

This integral cannot be expressed in terms of elementary functions. However, some information can be extracted analytically. Setting $\varepsilon=0$ one obtains the qOE density of states at the center of the band

$$\rho(0;\mu) = \frac{N^{3/2}}{\pi}\frac{\Gamma[(\mu+1)/2]}{\Gamma[\mu/2]}\sqrt{\frac{2}{\mu}}. \tag{37}$$

The behavior for large ε can be easily recognized by making the change of variables $2r=\sqrt{N}\varepsilon z$ in (36), which leads to

$$\rho(\varepsilon;\mu) \propto \varepsilon^{-(1+\mu)}\int_1^{\infty} dz \frac{\sqrt{z^2-1}}{z^{\mu+3}}\exp\left(-\frac{2\mu}{Nz^2\varepsilon^2}\right). \tag{38}$$

Evidently the tails vanish as $\varepsilon^{-(1+\mu)}$. This is also the behavior observed by Cizeau and Bouchaud in their "Lévy ensembles" of matrices having independent entries distributed according to the same law of Eq. (33) [11]. We note, however, that the analogies cannot be pushed further because our ensembles are rotationally invariant and Lévy ensembles are not (the ensembles of Ref. [11] belong to the so-called α-symmetric class [16]; see also Ref. [23]).

The calculation of the spacing distribution proceeds as before. We have to insert Eqs. (30) and (35) into (26). The result is

$$p(s;\mu) \propto s\int_0^\infty dr\, r^{-(\mu+3)} \exp\left[-\frac{\mu}{2r^2}\left(1+\frac{N^3 s^2}{2\pi\mu}\right)\right]. \tag{39}$$

The dependence on s can be easily isolated by a change of variables, so we can write

$$p(s;\mu) \propto s\left(1+\frac{N^3 s^2}{2\pi\mu}\right)^{-(1+\mu/2)}, \tag{40}$$

or alternatively

$$p(s;\mu) \propto s\, \exp_{q_s}(-\alpha s^2), \tag{41}$$

where

$$q_s \equiv \frac{\mu+4}{\mu+2} \quad \text{and} \quad \alpha \equiv \frac{N^3}{4\pi}\frac{\mu+2}{\mu}. \tag{42}$$

The function of Eq. (40) [or Eq. (41)] is *identical* in shape to the exact level-spacing distribution of the 2×2 qOE having the same μ (see the Appendix). Then, both distributions can be collapsed by a simple scaling of the arguments. This curious result constitutes a generalization of Wigner's surmise to qOE.

Remark. When analyzing spectral statistics it is usual to normalize energies so that the (local) average spacing is 1 (the spectrum is "unfolded"). This amounts to measuring energies in units of

$$\Delta \equiv \int_0^\infty ds\, s p(s). \tag{43}$$

Note, however, that in qOE the first moment of $p(s)$ does not exist for $\mu \leqslant 1$. In these cases, instead of Δ, one may alternatively use the energy scale

$$\tilde{\Delta} \equiv \left[\int_0^\infty ds\, s^{-1} p(s)\right]^{-1}. \tag{44}$$

Due to level repulsion there is no singularity at $s=0$, and $\tilde{\Delta}$ always exists, thus representing a characteristic energy of qOE. It is close to the inverse of the level density at $\varepsilon=0$

$$\tilde{\Delta} = \frac{2}{\pi\rho(0;\mu)}, \tag{45}$$

with $\rho(0;\mu)$ given in (37).

IV. NUMERICAL RESULTS

When thought of as clouds in R^d, via the map of Eq. (10), both ensembles qOE and GOE are spherically symmetric. This means that qOE can be constructed just by rescaling the radii of all points in the GOE cloud [16,17]. Thus, the construction of a qOE matrix $\mathbf{H}_1$ (with parameters q, σ, N) can be done in three steps. (i) Construct a GOE matrix $\mathbf{H}_0$ of size $N\times N$. In this case matrix elements are independent and can be calculated using Eqs. (17) and (18) with $q=1$. The radius of $\mathbf{H}_0$ is

$$r_0 = \sqrt{\mathrm{tr}\, \mathbf{H}_0^2}. \tag{46}$$

(ii) Choose a radius r_1 randomly according to the radial probability distribution $\mathcal{P}(r_1,q,\sigma,N)$ of Eq. (12). (iii) Define $\mathbf{H}_1$ as

$$\mathbf{H}_1 = \mathbf{H}_0 \frac{r_1}{r_0}. \tag{47}$$

This is the recipe we followed for constructing qOE matrices. (If, instead of being a random variable, r_1 is fixed, we obtain a matrix belonging to FTE.) The only difficulty is to devise the random number generator, especially when $\mathcal{P}(r)$ has very long tails. For this purpose we used a combination of the *rejection method* and the *transformation method*, as explained in Ref. [24].

In Fig. 2 we show histograms representing densities of states obtained from diagonalization of qOE matrices. It is clear that they are very well described by the formula (38), which was evaluated by direct numerical integration.

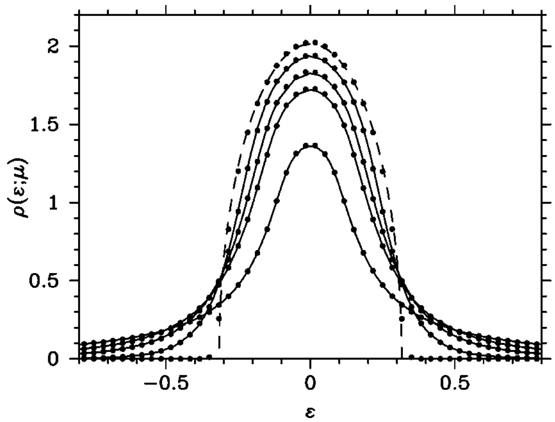

FIG. 2. Density of states in the ensembles qOE (normalized to 1). We compare histograms generated numerically (dots) with the theoretical result of Eq. (38) (curves). Each histogram was generated from a set of 10^5 matrices. We used the following values of μ: 0.5, 1.5, 2.5, 6.0, ∞. Densities with larger μ's have larger values at $\varepsilon=0$ and decay faster. In all cases $N=40$ and $\sigma=1$. The dashed line corresponds to the GOE semicircle ($N\to\infty$).

The statistics of level spacings is exhibited in Fig. 3. Histograms were obtained by binning data from 10^5 matrices. Each matrix contributed, with the "central" spacing between levels $\varepsilon_{N/2}$ and $\varepsilon_{N/2+1}$. The analytical curves are the q distributions of Eq. (40). Again, the agreement between theory and simulations is satisfactory. The values of μ were chosen in accordance with the following criterion. When $\mu=0.5$ all integer moments diverge. For $\mu=1.5$ ($\mu=2.5$) the first (second) moment exists but higher ones diverge. The case $\mu=6.0$ is intended to represent an ensemble qOE approaching GOE.

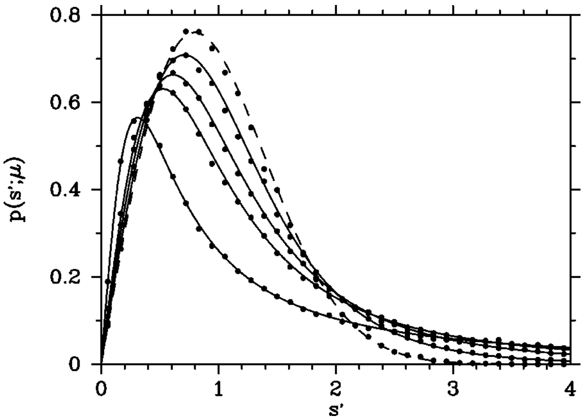

FIG. 3. Level-spacing distribution in the ensembles qOE. We compare histograms generated numerically (dots) with the theoretical result of Eq. (40) (curves). Each histogram was generated from a set of 10^5 matrices, each matrix contributing one pair of levels. We used the same values of μ as in Fig. 1. Distributions with larger μ's have higher maxima and decay faster. In all cases $N=40$ and $\sigma=1$. Spacings are measured in units of $1/\rho(0,\mu)$, i.e., $s' = s\rho(0,\mu)$ [see Eq. (37)].

V. CONCLUSIONS

This paper explores the possibility of using a modified maximum entropy method to construct random matrix ensembles. Whereas the usual logarithmic entropy leads to ensembles characterized by exponential laws, the power-law entropy of Eq. (4) naturally produces ensembles with long-tailed distributions (e.g., the so-called q Gaussians [13]). These ensembles (qOE) recover the Cauchy-Lorentz and Student's ones for special choices of the parameters.

Of course, the same families of ensembles can also be obtained from, say, the standard maximum entropy approach, but at the expense of introducing a complicated constraint. A similar situation arises in the maximum entropy approach to anomalous diffusion, where one can choose between an unappealing constraint [25] and a nonextensive entropy [18,21]. By sweeping the parameter q, one switches from sub- to superdiffusive regimes, $q=1$ giving normal diffusion [18,21].

In the present case, the entropic index q controls the confinement, allowing access to different universality classes. The anomalies we found can be associated with an effect of soft- (or weak) confinement. However, the confining potential in qOE is many-body, as opposed to the more common single-particle confinement in standard random matrix theories.

Gaussian ensembles are *ergodic* [26]. In Sec. III we explicitly used the fact that individual large qOE matrices have GOE statistics. This implies that qOE is *not ergodic* (if μ is finite). In random matrix theory, nonergodicity is considered to be a drawback because, it is argued, predictions (ensemble averages) are compared with data obtained from a single system. We do not object to this reasoning, but just mention that in some cases empirical data are indeed extracted from ensembles of Hamiltonians [27,28].

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APPENDIX: GENERALIZED WIGNER'S SURMISE

Here, we apply (26) to the level-spacing distribution in qOE with $N=2$. We want to calculate

$$p(s;q,\sigma) \equiv \langle \delta[(\varepsilon_2 - \varepsilon_1)(\mathbf{H}) - s] \rangle_{\text{qOE}}, \tag{A1}$$

where $\varepsilon_1 < \varepsilon_2$ are the eigenvalues of $\mathbf{H}$. First, we need $p(s;r)$ for FTE. This is a known result [17]

$$p(s;r) = \frac{s}{\sqrt{2}r} \frac{1}{\sqrt{2r^2 - s^2}} \tag{A2}$$

(if the argument of the root is positive, zero otherwise). According to (26), $p(s;q,\sigma)$ is obtained by averaging (A2) with the radial weight of qOE, Eq. (12) with $d=3$. We show the result for the case $q>1$

$$p(s;q,\sigma) \propto s \exp_{q''}(-\lambda'' s^2), \tag{A3}$$

where

$$q'' = \frac{1+q}{3-q}, \tag{A4}$$

and

$$\lambda'' = \frac{3-q}{4\sigma^2(5/3 - q)}. \tag{A5}$$

Equation (A3) plays the role of Wigner's conjecture for qOE. It is more useful to rewrite (A3) in terms of the parameter μ of Eq. (14). The result is

$$p(s;q,\sigma) \propto s\left(1 + \frac{3s^2}{2\mu}\right)^{-(1+\mu/2)}. \tag{A6}$$

In Sec. III this expression is compared with the spacing distribution for the large-N case [Eq. (40)].

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